A Probabilistic Weyl-law for Randomly Perturbed Berezin-Toeplitz Operators

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Preliminaries – Brief Version

1. **Goal:** For each $f \in C^\infty(M)$, M a symplectic manifold, want to get a family of finite rank operators, $T_N f$, indexed by $N \in \mathbb{Z}_{>0}$ on a Hibert space (classical to quantum).
2. **Basic idea:** On M get a weighted L^2 space **depending on** N. Find a finite dimensional subspace, and let Π_N be the orthogonal projection onto this smaller space.
3. The **Toeplitz operartor** associated to f, is $T_N f := \Pi_N \circ M_f \circ \Pi_N$ (where M_f is multiplication by f), which is a finite rank operator.

Theorem 1: Main Result

Given:

- $(M, \sigma) :=$ *compact, connected,* d*–dimensional Kähler manifold with a holomorphic line bundle* L *with positively curved Hermitian metric, with* **volume form** $\mu_d := \sigma^{\wedge d}/d!$.
- $f \in C^\infty(M; \mathbb{C})$ *such that* $\exists\, \kappa \in (0,1]$ *such that* $\mu_d(x \in M : |f(x) - z|^2 \le t) = \mathcal{O}(t^\kappa)$.
- $\mathcal{G} : H^0(M, L^N) \to H^0(M, L^N)$ *whose matrix representation with respect to a fixed basis is a random matrix whose entries are* **i.i.d. complex Gaussian random variables** *with mean* 0 *and variance* 1.

Then for any **open** $\Omega \subset \mathbb{C}$:

$$\left(\frac{2\pi}{N}\right)^d \#\{\mathrm{Spec}(T_N f + N^{-d}\mathcal{G}) \cap \Omega\} \xrightarrow{N\to\infty} \mu_d(x \in M : f(x) \in \Omega) \qquad (1)$$

almost surely.

Preliminaries – Slightly Longer

These preliminaries can be found in Le Floch's textbook *A Brief Introduction to Berezin–Toeplitz Operators on Compact Kähler Manifolds* [Le Floch 2018].

1. A **Kähler manifold** is a complex manifold M with with three compatible structures: a complex structure, a symplectic form σ, and a Riemannian metric.
2. We assume on M there is a **holomorphic line bundle** L with a positively curved Hermitian metric h. Locally, on each trivialization, there exists a smooth, real-valued function φ such that over each fiber $x \in M$,
$$h(u, v) = e^{-\varphi(x)} u\bar{v}. \qquad (2)$$
3. The locally defined φ is called a **Kähler potential**, and is related to the globally defined sympletic form σ by $i\partial\bar{\partial}\varphi = \sigma$.
4. We let L^N be the N^{th} tensor power of the line bundle L which has Hermitian metric h^N.
5. We define an inner product of smooth sections on L^N by defining
$$\langle u, v\rangle = \int_M u\bar{v}e^{-N\varphi}\left(\frac{\sigma^{\wedge d}}{d!}\right) \qquad (3)$$
where d is the complex dimension of M.
6. Let $L^2(M, L^N)$ be the space of smooth sections of L^N which have finite L^2 norm. Let $H^0(M, L^N)$ be the space of holomorphic sections in $L^2(M, L^N)$. This turns out to be a finite dimensional space.
7. Let Π_N be the orthogonal projection from $L^2(M, L^N)$ to $H^0(M, L^N)$, this is called the **Bergman projector**
8. Given a smooth function f on X, the **Toeplitz operator** $T_N f$ acts on holomorphic sections $u \in H^0(M, L^N)$ by
$$T_N f(u) = \Pi_N(fu). \qquad (4)$$
9. For each $N \in \mathbb{N}$, $T_N f$ is a finite rank operator mapping $H^0(M, L^N)$ to itself.

The same result for a different quantization – Vogel's Theorem

Theorem 1 is a generalization of a theorem proven by Martin Vogel [Vogel 2020]. Vogel studied a different, but related, quantization (Weyl-quantization of periodic functions restricted to a finite vector space of Dirac distributions). In this case, functions on the torus of real dimension $2d$ are associated to $N^d \times N^d$ matrices. As an example, the function $f(x, \xi) = \cos(x) + i\cos(\xi)$ is associated to the family of matrices

$$f_N = \begin{pmatrix} \cos(2\pi/N) & i/2 & 0 & 0 & \cdots & i/2 \\ i/2 & \cos(4\pi/N) & i/2 & 0 & \cdots & 0 \\ 0 & i/2 & \cos(6\pi/N) & i/2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & i/2 & \cos(2(N-1)\pi/N) & i/2 \\ i/2 & 0 & \cdots & 0 & i/2 & \cos(2\pi) \end{pmatrix}. \qquad (5)$$

Below is a plot of the spectrum of f_N and f_N with a small random perturbation.

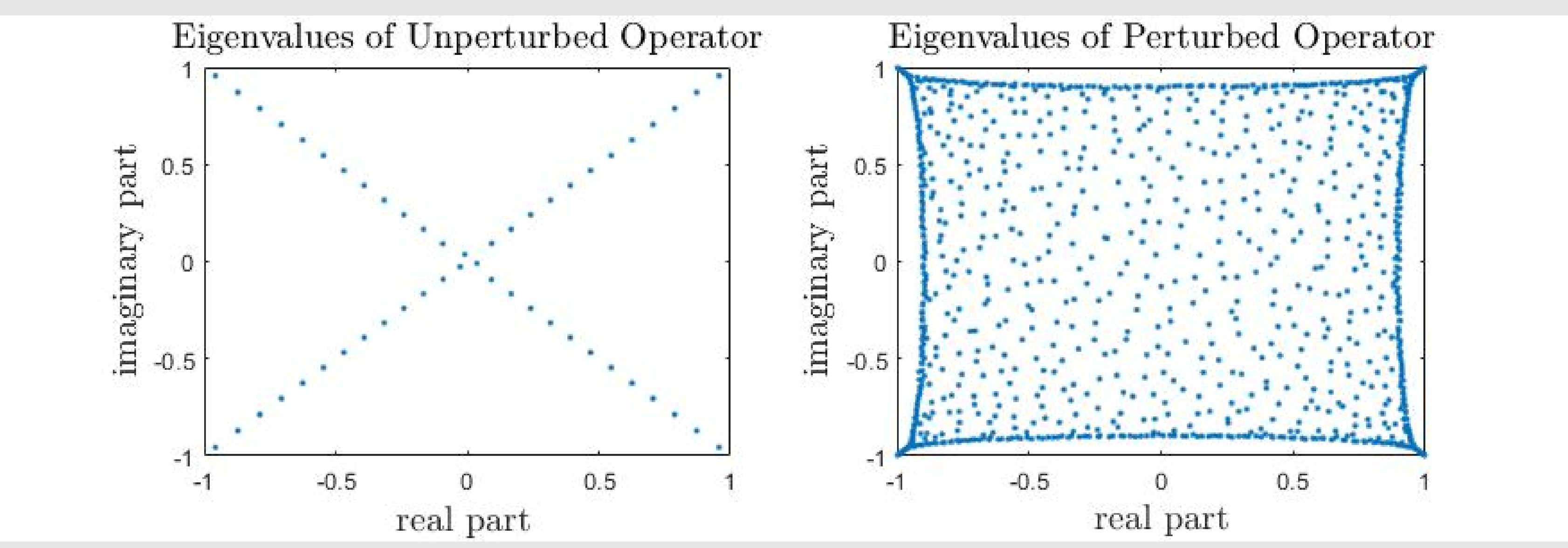

Figure 1. Left is f_N with no perturbation (this is known as the Scottish flag operator) ($N = 50$). Right is f_N with a small random perturbation ($N = 1000$).

A consequence [Vogel 2020] is that the in any region $\Omega \subset \mathbb{C}$

$$N^{-1}\#(\mathrm{Spec}(f_N + N^{-d}\mathcal{G}) \cap \Omega) \xrightarrow{N\to\infty} |\{(x,\xi) \in \mathbb{T}^{2d} : \cos(x) + i\cos(\xi) \in \Omega\}| \qquad (6)$$

almost surely.

Figure 2. Animations of these numerics, as well as others, can be found by scanning the above QR code.

Example: $\mathbb{CP}^1$

Consider $\mathbb{CP}^1$ (complex projective space of complex dimension 1). This is a Kähler manifold (for the symplectic form σ below). If we take the tautological line bundle, and take the dual, then we get a positively curved Hermitian line bundle, for which we can take tensor powers. We can consider a single chart, in which case we can write:

1. $\sigma(x) = (1 + |x|^2)^{-2}\,\mathrm{d}x \wedge \mathrm{d}\bar{x}$ (symplectic form),
2. $\varphi(x) = \log(1 + |x|^2)$ (Kähler potential),
3. Smooth sections can be identified with smooth functions on $\mathbb{C}$ on which we have the L^2 inner product
$$\langle f, g\rangle_{L^2(M,L^N)} = \int_{\mathbb{C}} \frac{f(z)\overline{g(z)}}{(1+|z|^2)^{N+2}}\,\mathrm{d}m(z). \qquad (7)$$
4. Then $H^0(X, L^N)$ are polynomials of degree $\le N$ which has the orthonormal basis
$$\left\{ \underbrace{\sqrt{\binom{N}{k}\frac{N+1}{\pi}}\, z^k}_{:=e_k(z)} : k = 0, \dots, N \right\}. \qquad (8)$$
5. The Bergman projector has kernel
$$\Pi_N(z, w) = \sum_{k=0}^N e_k(z)\overline{e_k(w)} = \cdots = \frac{N+1}{\pi}(1 + z\bar{w})^N. \qquad (9)$$
6. If f is a smooth function on $\mathbb{CP}^1$, then in coordinates, if $u \in H^0(M, L^N)$:
$$T_N f(u)(z) = \int_{\mathbb{C}} \Pi_N(z, w) f(w) u(w) \frac{\mathrm{d}m(w)}{(1+|w|^2)^{N+2}}. \qquad (10)$$

Numerics for $\mathbb{CP}^1$

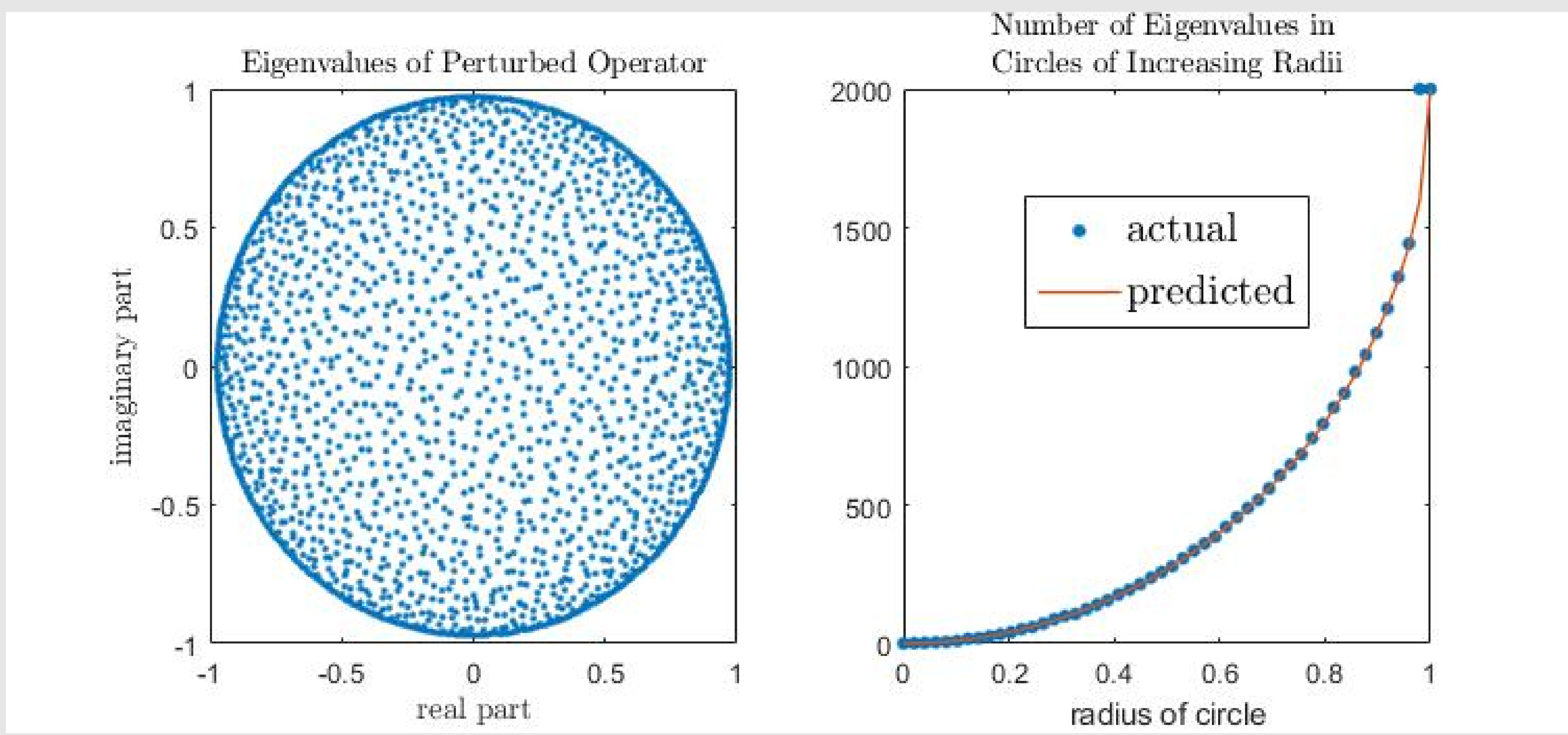

Figure 3. Identifying $\mathbb{CP}^1$ with the 2-sphere in $\mathbb{R}^3$ with coordinates x_1, x_2, x_3, we consider the quantization of $f = x_1 + ix_2$. On the left is the spectrum of $T_N f$ for $N = 2000$ with a random perturbation. On the right is the predicted number of eigenvalues within circles centered at the origin of increasing radii plotted against the predicted number of eigenvalues from Theorem 1.

Sketch Of Proof

1. Using a Borel-Cantelli argument, and **logarithmic potentials**, it is enough to show
$$\sum_{N=1}^\infty \mathbb{P}(|N^{-d}\log|\det(T_N f + N^{-d}\mathcal{G})| - \int_M \log|z - f(x)| d\mu_d(x)| > N^{-\gamma}) < \infty \qquad (11)$$
for some $\gamma > 0$
2. Set up a **Grushin problem** (also referred to as **Shur compliment formula**) to control this.
3. Main obstruction is to estimate $\psi(N^\delta T_N f)$ as a Toeplitz operator for $\delta \in (0,1)$ for some $\psi \in C_0^\infty(\mathbb{R}; [0,1])$.
4. Require developing an **exotic calculus** for Toeplitz operators.

Exotic Calculus

1. Consider the Kähler manifold $\mathbb{C}$. In this case define
$$S_\delta(m) = \{f \in C^\infty(\mathbb{C}) : |\partial^\alpha f| \le C_\alpha N^{\delta|\alpha|} m\} \qquad (12)$$
for $\delta \in [0, 1/2)$ and m an order function.
2. Need to show that if $f \in S_\delta(m_1)$ and $g \in S_\delta(m_2)$ then there exists $h \in S_\delta(m_3)$ such that
$$T_N f \circ T_N g = T_N h + \mathcal{O}(N^{-\infty}). \qquad (13)$$
3. The main idea is to obtain an asymptotic expansion of the Schwartz kernel of the Toeplitz operators with appropriate remainder.
4. This involves applying Melin and Sjöstrand's method of **complex stationary phase** [Melin and Sjöstrand 1975]. This involves almost analytically extending functions in $S_\delta(m)$ which obey weaker $\bar{\partial}$ estimates. This has to be be carefully handled to obtain the correct asymptotic expansion of the Schwartz kernel.

References

Vogel, M. (2020). "Almost Sure Weyl Law for Quantized Tori". In: *Communications in Mathematical Physics.*

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